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## Representation of a Differential System\*

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## 1. INTRODUCTION

Starting with a general formulation of a differential system (DS), two other system representations are derived, and the relationships between the three representations are investigated. The first, called a generalized dynamical system (GDS), studied by Roxin [1, 2, 3], is defined via the "attainability function" of the DS. Following Warga [4], a relaxed differential system (RDS) is defined by augmenting the set of "permissible" velocities of the DS. It is shown that the set of solutions of the derived GDS and the set of solutions of the derived RDS coincide with the closure of the set of trajectories of the DS in the topology of uniform convergence over finite intervals. In a less general and somewhat different setting, Warga [4], has obtained this relationship between a DS and its derived RDS. The differences between Warga's formulation and this one is the replacement of continuity a.e. with respect to the time parameter (required by Warga) by local integrability. Furthermore, the part played by the "control variable" of the DS is made more explicit.

## 2. NOTATIONS AND DEFINITIONS

2.1. If  $x$  is a vector in  $n$ -dimensional real Euclidean space  $R^n$ , the norm of  $x$  is denoted by

$$|x| = \left( \sum_{i=1}^n x_i^2 \right)^{1/2}.$$

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2.2.  $\mathcal{C}[t_0, t_1]$  denotes the real Banach space of all continuous functions  $x: [t_0, t_1] \rightarrow R^n$  where the norm of  $x$  is given by

$$\|x\| = \max_{t_0 \leq t \leq t_1} |x(t)|.$$

2.3. If  $A$  is a subset of a topological space,  $\bar{A}$  denotes its closure.

2.4 If  $x \in R^n$  and  $A$  is a subset of  $R^n$ , then

$$\rho(A, x) = \rho(x, A) = \inf\{|x - y| \mid y \in A\};$$

and for  $\epsilon > 0$

$$S_\epsilon(A) = \{x \in R^n \mid \rho(x, A) < \epsilon\}.$$

### 3. THE DIFFERENTIAL SYSTEM (DS)

The differential system (DS) under consideration is described by the vector differential equation

$$\dot{x}(t) = f(x(t), t, u(t))$$

where the following conditions hold:

(i)  $t \in R^1$ ,  $x(t) \in R^n$  and  $u(t) \in U$  for each  $t$ , where  $U$  is an arbitrary fixed, nonempty subset of  $R^m$ .

(ii) For each fixed  $t$ ,  $f$  is continuous in  $(x, u)$  for all  $(x, u)$  in  $R^n \times R^m$ . And for each fixed  $(x, u)$ ,  $f$  is measurable with respect to  $t$ .

(iii) There is a locally-integrable function  $k$ , such that for every  $u$  in  $U$  and  $x, x'$  in  $R^n$ ,

$$|f(x, t, u) - f(x', t, u)| \leq k(t) |x - x'|.$$

(iv) There are a locally-integrable function  $\ell$  and positive numbers  $M$  and  $N$  such that for every  $x$  in  $R^n$  and  $u \in U$ ,

$$|f(x, t, u)| \leq \ell(t)(M + N |x|).$$

DEFINITION 3.1. A function  $u: [t_0, t_1] \rightarrow R^m$  is called an *admissible control*, if it is measurable, and if  $u(t) \in U$  for each  $t \in [t_0, t_1]$ .

DEFINITION 3.2. A function  $x: [t_0, t_1] \rightarrow R^n$  is called a *trajectory* of the DS, if there exists an admissible control  $u$ , such that  $x$  is absolutely continuous and  $\dot{x}(t) = f(x(t), t, u(t))$  a.e. in  $[t_0, t_1]$ . The trajectory is said to start at  $(x(t_0), t_0)$  and end at  $(x(t_1), t_1)$ .

DEFINITION 3.3. (i) For  $x_0$  in  $R^n$ ,  $t_0, t_1$  in  $R^1$  with  $t_0 \leq t_1$ , let  $T(x_0, t_0, t_1)$  be the set of those vectors  $x$  in  $R^n$  for which there exists a trajectory of the DS starting at  $(x_0, t_0)$  and ending at  $(x, t_1)$ . In other words,  $T(x_0, t_0, t_1)$  is the set of states attainable at time  $t_1$  by the DS starting at  $(x_0, t_0)$  and using an admissible control.

(ii) Let  $Q(x_0, t_0, t_1) = \overline{T(x_0, t_0, t_1)}$ .  $Q$  is called the *attainability function* of the generalized dynamical system (GDS) derived from the DS. (See Section 5)

DEFINITION 3.4. The set of all trajectories defined on  $[t_0, t_1]$  and starting at  $(x_0, t_0)$  is denoted by  $\mathcal{T}(x_0, t_0, t_1)$ .  $\mathcal{T}(x_0, t_0, t_1)$  is considered a subset of  $\mathcal{C}[t_0, t_1]$ .

LEMMA 3.1. Let  $x_0 \in R^n$ ;  $t_0, t_1 \in R^1$  with  $t_0 \leq t_1$ . Then

$$\bigcup_{t_0 \leq t \leq t_1} (T(x_0, t_0, t))$$

is a bounded subset of  $R^n$ , and hence  $\mathcal{T}(x_0, t_0, t_1)$  is a bounded subset of  $\mathcal{C}[t_0, t_1]$ .

*Proof.* Let  $t \in [t_0, t_1]$ . Each point of  $T(x_0, t_0, t)$  is given by

$$x(t) = x_0 + \int_{t_0}^t f(x(s), s, u(s)) ds$$

for some admissible control  $u$ . From condition (iv) of the DS we have

$$|x(t)| \leq |x_0| + \int_{t_0}^t Ml(s) ds + \int_{t_0}^t Nl(s) |x(s)| ds,$$

so that by Gronwall's lemma [5, p. 11]

$$\begin{aligned} |x(t)| &\leq |x_0| \exp \left( \int_{t_0}^t Nl(s) ds \right) + \int_{t_0}^t \exp \left( \int_s^t Nl(\tau) d\tau \right) Ml(s) ds \\ &= \varphi(x_0, t_0, t). \end{aligned}$$

Clearly  $\varphi(x_0, t_0, t)$  is a continuous monotonic function of  $t$  and is independent of the admissible control  $u$  so that the result follows.

LEMMA 3.2. (i)  $\mathcal{T}(x_0, t_0, t_1)$  is an equicontinuous subset of  $\mathcal{C}[t_0, t_1]$ .

(ii)  $T(x_0, t_0, t_1)$  is jointly-continuous in all its arguments; i.e., given

$x_0$  in  $R^n$ , also  $t_0, t_1$  in  $R^1$  with  $t_0 \leq t_1$ , and  $\varepsilon > 0$  there is a  $\delta > 0$  such that if  $|x_0 - x'_0| < \delta, |t_0 - t'_0| < \delta, |t_1 - t'_1| < \delta$  and  $t'_0 \leq t'_1$ , then

$$T(x_0, t_0, t_1) \subset S_\varepsilon[T(x'_0, t'_0, t'_1)],$$

and

$$T(x'_0, t'_0, t'_1) \subset S_\varepsilon[T(x_0, t_0, t_1)].$$

*Proof.* Consider two trajectories  $x$  and  $x'$  starting at  $(x_0, t_0)$  and  $(x'_0, t'_0)$ , respectively, and determined by the same admissible control  $u$  defined for  $t \geq \min\{t_0, t'_0\}$ . Thus,

$$x(t) = x_0 + \int_{t_0}^t f(x(s), s, u(s)) ds,$$

whereas

$$x'(t') = x'_0 + \int_{t'_0}^{t'} f(x'(s), s, u(s)) ds.$$

Let

$$\begin{aligned} \Delta x(t, t') &= |x(t) - x'(t')| \leq |x(t) - x(t')| + |x(t') - x'(t')| \\ &= \Delta x_1(t, t') + \Delta x_2(t'). \end{aligned} \quad (3.1)$$

To prove (i), let  $B$  be a finite bound of  $\mathcal{F}(x_0, t_0, t_1)$ . Such a bound exists by Lemma 3.1. Then for  $t_0 \leq t \leq t' \leq t_1$ ,

$$\begin{aligned} \Delta x_1(t, t') &= |x(t) - x(t')| \leq \int_t^{t'} |f(x(s), s, u(s))| ds \\ &\leq \int_t^{t'} (M + NB) l(s) ds. \end{aligned} \quad (3.2)$$

Since  $l$  is locally-integrable, given  $\varepsilon > 0$ , there is a  $\delta_1 > 0$  such that if  $|t - t'| < \delta_1$  then

$$\int_t^{t'} (M + NB) l(s) ds < \frac{\varepsilon}{2}. \quad (3.3)$$

Since the admissible control  $u$  was arbitrary, we see from (3.2) and (3.3) that for every  $x$  in  $\mathcal{F}(x_0, t_0, t_1)$  and for every  $t, t'$  in  $[t_0, t_1]$  with  $|t - t'| < \delta_1$ ,

$$|\Delta x_1(t, t')| < \frac{\varepsilon}{2}. \quad (3.4)$$

and (i) is proved.

To prove (ii), consider

$$\begin{aligned}
 \Delta x_2(t'_1) &= |x(t'_1) - x'(t'_1)| \leq |x_0 - x'_0| + \left| \int_{t_0}^{t'_1} f(x(s), s, u(s)) ds \right. \\
 &\quad \left. - \int_{t'_0}^{t'_1} f(x'(s), s, u(s)) ds \right|, \\
 &\leq |x_0 - x'_0| + \int_{t'_0}^{t'_1} |f(x(s), s, u(s)) - f(x'(s), s, u(s))| ds \\
 &\quad + \int_{t_0}^{t'_0} |f(x(s), s, u(s))| ds, \\
 &\leq |x_0 - x'_0| + \int_{t'_0}^{t'_1} k(s) |x(s) - x'(s)| ds + \int_{t_0}^{t'_0} (M + NB) l(s) ds.
 \end{aligned} \tag{3.5}$$

Let

$$\beta = \frac{1}{2} \epsilon \left( \exp \int_{t'_0}^{t'_1} k(s) ds \right)^{-1}.$$

Let  $\delta_2 > 0$  be such that if  $|t_0 - t'_0| < \delta_2$ , then

$$\int_{t_0}^{t'_0} (M + NB) l(s) ds < \frac{1}{2} \beta.$$

Then if  $|t_0 - t'_0| < \delta_2$  and  $|x_0 - x'_0| < \frac{1}{2}\beta$  we have, from (3.5),

$$\Delta x_2(t'_1) \leq \beta + \int_{t'_0}^{t'_1} k(s) |x(s) - x'(s)| ds.$$

By Gronwall's lemma [5, p. 11],

$$\Delta x_2(t'_1) \leq \beta \exp \left( \int_{t'_0}^{t'_1} k(s) ds \right) \leq \frac{1}{2} \epsilon. \tag{3.6}$$

Substituting (3.6) and (3.4) into (3.1), we see that for  $\delta = \min\{\delta_1, \delta_2, \frac{1}{2}\beta\}$ ,  $\Delta x(t_1, t'_1) \leq \epsilon$  and hence (ii) is proved.

#### 4. THE GENERALIZED DYNAMICAL SYSTEM (GDS)

Following Roxin [3], we define a GDS by an "attainability function"  $F(x_0, t_0, t_1)$  which, for every  $x_0$  in  $R^n$  and  $t_0, t_1$  in  $R^1$  with  $t_0 \leq t_1$ , represents

the set of states attainable by the system at time  $t_1$ , starting in state  $x_0$  at time  $t_0$ . The attainability function  $F$  satisfies the following axioms:

4.A1. For  $x_0$  in  $R^n$  and  $t_0, t_1$  in  $R^1$  with  $t_0 \leq t_1$ ,  $F(x_0, t_0, t_1)$  is a nonempty, compact subset of  $R^n$ .

4.A2. For  $x_0$  in  $R^n$  and  $t_0$  in  $R^1$ ,  $F(x_0, t_0, t_0) = \{x_0\}$ .

4.A3.  $F(x_0, t_0, t_1)$  is jointly-continuous in all its arguments, i.e., given  $x_0$  in  $R^n$ , also  $t_0, t_1$  in  $R^1$  with  $t_0 \leq t_1$ , and  $\epsilon > 0$  there is a  $\delta > 0$  such that if  $|x_0 - x'_0| < \delta$ ,  $|t_0 - t'_0| < \delta$ ,  $|t_1 - t'_1| < \delta$ , and  $t'_0 \leq t'_1$ , then

$$F(x_0, t_0, t_1) \subset S_\epsilon[F(x'_0, t'_0, t'_1)],$$

and

$$F(x'_0, t'_0, t'_1) \subset S_\epsilon[F(x_0, t_0, t_1)].$$

4.A4. For  $x_0$  in  $R^n$  and  $t_0, t_1, t_2$  in  $R^1$  with  $t_0 \leq t_1 \leq t_2$

$$F(x_0, t_0, t_2) = \bigcup_{x \in F(x_0, t_0, t_1)} F(x, t_1, t_2).$$

DEFINITION 4.1. A function  $x : [t_0, t_1] \rightarrow R^n$  is called a *motion* of the GDS if, for  $t_0 \leq \tau_0 \leq \tau_1 \leq t_1$ ,

$$x(\tau_1) \in F(x(\tau_0), \tau_0, \tau_1).$$

The motion is said to start at  $(x(t_0), t_0)$  and end at  $(x(t_1), t_1)$ .

Remark 4.1. From axiom 4.A4 it is clear that every motion is a continuous function of time.

DEFINITION 4.2. The set of motions of the GDS defined on  $[t_0, t_1]$  and starting at  $(x_0, t_0)$  is denoted by  $\mathcal{M}(x_0, t_0, t_1)$ .  $\mathcal{M}(x_0, t_0, t_1)$  is considered a subset of  $\mathcal{C}[t_0, t_1]$ .

The following lemma is proved in Roxin [3]:

LEMMA 4.1.  $\mathcal{M}(x_0, t_0, t_1)$  is a closed subset of  $\mathcal{C}[t_0, t_1]$ .

## 5. THE DERIVED GDS

The function  $Q$  of Definition 3.3 (ii) is used to derive a GDS from the given DS.

THEOREM 5.1. The function  $Q$  of Definition 3.3(ii) satisfies the axioms 4.A1–4.A4 of an attainability function for a GDS.

*Proof.* Conditions (i)–(iv) of Section 3 guarantee the existence of a solution to the DS for every initial condition  $(x_0, t_0)$  and every admissible control. Therefore  $T(x_0, t_0, t_1)$  is nonempty. By Lemma 3.1,  $T(x_0, t_0, t_1)$  is bounded so that  $Q(x_0, t_0, t_1)$  is nonempty and compact. Hence  $Q$  satisfies 4.A1.

For a DS it is evident that  $T(x_0, t_0, t_0) = \{x_0\}$  so that  $Q(x_0, t_0, t_0) = \{x_0\}$ .

By Lemma 3.2,  $T$  is jointly-continuous in all its arguments. Since  $Q$  is the closure of  $T$ , it follows that  $Q$  is also jointly-continuous and hence satisfies 4.A3.

From the definition of the function  $T$  it is clear that for  $x_0$  in  $R^n$ , and  $t_0, t_1, t_2$  in  $R^1$  with  $t_0 \leq t_1 \leq t_2$

$$T(x_0, t_0, t_2) = \bigcup_{x \in T(x_0, t_0, t_1)} T(x, t_1, t_2). \quad (5.1)$$

By Lemma 3.2,  $T$  satisfies the hypothesis of Lemma A1 of the appendix, so that

$$\left\{ \overline{\bigcup_{x \in T(x_0, t_0, t_0)} T(x, t_1, t_2)} \right\} = \bigcup_{x \in \overline{T(x_0, t_0, t_1)}} \overline{T(x, t_1, t_2)}. \quad (5.2)$$

Then from (5.2) and Definition 3.3(ii)

$$Q(x_0, t_0, t_2) = \bigcup_{x \in Q(x_0, t_0, t_1)} Q(x, t_1, t_2).$$

This completes the proof of Theorem 5.1.

## 6. RELATIONSHIP BETWEEN MOTIONS AND TRAJECTORIES

Let  $a, b$  be fixed numbers with  $a \leq b$  and  $x_0$  a fixed element of  $R^n$ ,  $I = [a, b]$ ,  $\mathcal{M} = \mathcal{M}(x_0, a, b)$  be the set of motions of the derived GDS defined on  $[a, b]$  and starting at  $(x_0, a)$ , and  $\mathcal{F} = \mathcal{F}(x_0, a, b)$ .

**THEOREM 6.1.**  $\mathcal{M} = \mathcal{F}$ .

*Proof.* From the definitions of a trajectory and a motion, and from the definitions of  $Q$ , it is evident that  $\mathcal{M} \supset \mathcal{F}$ . By Lemma 4.1  $\mathcal{M}$  is closed, so that

$$\mathcal{M} = \bar{\mathcal{M}} \supset \mathcal{F}. \quad (6.1)$$

In order to show inclusion in the converse direction it will be demonstrated that for every motion  $x$  in  $\mathcal{M}$ , there is a trajectory  $y \in \mathcal{F}$  arbitrarily close to  $x$ .

Let  $\epsilon > 0$  be given. By Lemma 3.2  $\mathcal{T}$  is equicontinuous so that there is  $\delta_1 > 0$  such that if  $t, t'$  are in  $I$  with  $|t - t'| < \delta_1$ , then

$$|y(t) - y(t')| < \epsilon/3 \quad (6.2)$$

for every  $y \in \mathcal{T}$ .

Let  $x \in \mathcal{M}$  be given. Then  $x$  is uniformly-continuous on  $I$  so that there is  $\delta_2 > 0$  such that if  $t, t'$  are in  $I$  with  $|t - t'| < \delta_2$ , then

$$|x(t) - x(t')| < \epsilon/3. \quad (6.3)$$

Letting  $\delta = \min\{\delta_1, \delta_2\}$ , we choose a finite sequence  $a = t_0 < t_1 < \dots < t_p = b$  such that  $t_k - t_{k-1} < \delta$  for each  $k$ . Denote  $x_k = x(t_k)$  for each  $k$ .

By 4.A3 there is  $\eta_{p-1} > 0$  such that if  $|z - x_{p-1}| < \eta_{p-1}$ , then

$$Q(x_{p-1}, t_{p-1}, t_p) \subset S_{\epsilon/3}[Q(z, t_{p-1}, t_p)]. \quad (6.4)$$

For  $0 \leq k \leq p-2$  there is  $\eta_k > 0$  such that if  $|z - x_k| < \eta_k$ , then

$$Q(x_k, t_k, t_{k+1}) \subset S_{r_k}[Q(z, t_k, t_{k+1})], \quad (6.5)$$

where  $r_k = \min\{\epsilon/3, \eta_{k+1}\}$ .

Now let  $y_1 \in T(x_0, t_0, t_1)$  such that  $|y_1 - x_1| < r_0$ , and for  $2 \leq k \leq p$  let  $y_k \in T(y_{k-1}, t_{k-1}, t_k)$  such that  $|y_k - x_k| < r_{k-1}$ . This is possible because of (6.5), (6.4), and the fact that  $Q$  is the closure of  $T$ . Since each  $y_k \in T(y_{k-1}, t_{k-1}, t_k)$ , there is a trajectory  $v_k$  starting at  $(y_{k-1}, t_{k-1})$  and ending at  $(y_k, t_k)$ . Also, for  $t_{k-1} \leq t \leq t_k$ ,

$$\begin{aligned} |x(t) - v_k(t)| &\leq |x(t) - x_k| + |x_k - y_k| + |y_k - v_k(t)| \\ &\leq \epsilon \end{aligned}$$

from (6.2), (6.3) and because  $|x_k - y_k| \leq r_k \leq (\epsilon/3)$ . Let  $y \in \mathcal{T}$  be defined by  $y(t) = v_k(t)$  for  $t_{k-1} \leq t \leq t_k$ . Then  $\|x - y\| \leq \epsilon$ . Since  $\epsilon > 0$  and  $x \in \mathcal{M}$  are arbitrary  $\mathcal{M} \subset \mathcal{T}$ , and the theorem is proved.

## 7. THE RELAXED DIFFERENTIAL SYSTEM (RDS)

Following Warga [4], we derive a relaxed differential system from the DS of Section 3. The class of differential systems under study is more general than that investigated by Warga. The difference lies in the replacement of continuity a.e. in  $t$  (required by Warga) by local integrability.



DEFINITION 7.1. For the DS of Section 3 let

$$F(x, t) = \{y \in R^n \mid y = f(x, t, u), u \in U\}.$$

Thus  $F(x, t)$  represents the set of "permissible velocities" of the DS at the phase  $(x, t)$ .

DEFINITION 7.2. For  $x$  in  $R^n$  and  $t$  in  $R^1$  let  $G(x, t)$  be the convex closure of  $F(x, t)$ .

DEFINITION 7.3. (i) A function  $x : [t_0, t_1] \rightarrow R^n$  is called a *relaxed trajectory* if  $x$  is absolutely-continuous and

$$\dot{x}(t) \in G(x(t), t)$$

a.e. in  $[t_0, t_1]$ . The relaxed trajectory is said to start at  $(x(t_0), t_0)$  and end at  $(x(t_1), t_1)$ .

(ii) The set of relaxed trajectories starting at  $(x_0, t_0)$  and defined on  $[t_0, t_1]$  is denoted by  $\mathcal{R}(x_0, t_0, t_1)$ .  $\mathcal{R}(x_0, t_0, t_1)$  is considered a subset of  $\mathcal{C}[t_0, t_1]$ .

Let  $a, b$  with  $a \leq b$  be fixed finite numbers and  $x_0$  a fixed vector in  $R^n$ . Let  $I = [a, b]$ ,  $\mathcal{R} = \mathcal{R}(x_0, a, b)$ , and  $\mathcal{T} = \mathcal{T}(x_0, a, b)$ .

Lemma 7.1 is standard, and Lemma 7.2 is proved in Cullum [6].

LEMMA 7.1. Let  $A$  be a closed convex subset of  $R^n$ . Then  $x^* \in A$  if and only if for each  $y \in R^n$

$$\inf_{x \in A} \langle y, x \rangle \leq \langle y, x^* \rangle \leq \sup_{x \in A} \langle y, x \rangle.$$

LEMMA 7.2. Let  $\varphi_m : I \rightarrow R^n$  for  $m = 1, 2, 3, \dots$ , be measurable functions (uniformly bounded by an integrable function),  $\varphi : I \rightarrow R^n$  be integrable on  $I$ , and suppose that for each  $y \in R^n$  and each measurable subset  $E$  of  $I$ ,

$$\int_E \langle y, \varphi_m(t) \rangle dt \rightarrow \int_E \langle y, \tilde{\varphi}(t) \rangle dt.$$

Then, for each  $y \in R^n$ ,

$$\overline{\lim}_m \langle y, \varphi_m(t) \rangle \geq \langle y, \tilde{\varphi}(t) \rangle \geq \underline{\lim}_m \langle y, \varphi_m(t) \rangle$$

a.e. in  $I$ .

LEMMA 7.3. Let  $x \in \mathcal{R}$  be fixed. Then, for each  $\epsilon > 0$  there exist measurable

functions  $\alpha_i, u_i$  for each  $i = 1, \dots, n + 1$  such that for each  $t$  in  $I$ ,  $\alpha_i(t) \geq 0$  with

$$\sum_i \alpha_i(t) = 1,$$

$u_i(t) \in U$  and

$$\int_a^t \left| \dot{x}(s) - \sum_i \alpha_i(s) f(x(s), s, u_i(s)) \right| ds \leq \epsilon. \quad (7.1)$$

*Proof.* Let  $q: I \rightarrow R^1$  be a measurable function. A point  $t$  in  $I$  is said to be a regular point of  $q$  if

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} |q(t) - q(s)| ds = 0.$$

We note that if  $q$  is in  $L_1$ , almost all points of  $I$  are regular points of  $q$ .

For  $t$  in  $I$  and  $u \in U$ , let  $g(t, u) = f(x(t), t, u)$ . Let  $U_0$  be a countable dense subset of  $U$ . For each  $u$  in  $U_0$ , let  $T_u$  be the set of regular points of the function  $g(t, u)$  and let  $T_x$  be the set of regular points of  $\dot{x}(t)$ . Let

$$T = T_x \cap \left\{ \bigcap_{u \in U_0} T_u \right\}.$$

Then  $\text{measure}(T) = \text{measure}(I)$ .

Let  $\gamma$  be any positive number. Then, for each  $t \in T$  there exists numbers  $\beta_i(t) \geq 0$ ,  $1 \leq i \leq n + 1$  with

$$\sum_i \beta_i(t) = 1,$$

and vectors  $v_i(t) \in U_0$ ,  $1 \leq i \leq n + 1$  such that

$$\left| \dot{x}(t) - \sum_i \beta_i(t) g(t, v_i(t)) \right| < \gamma. \quad (7.2)$$

Furthermore, there is a number  $h(t) > 0$  such that for  $0 < h < h(t)$ ,

$$\frac{1}{h} \int_t^{t+h} |\dot{x}(t) - \dot{x}(s)| ds < \gamma, \quad (7.3)$$

and

$$\frac{1}{h} \int_t^{t+h} |g(t, v_i(t)) - g(s, v_i(t))| ds < \gamma. \quad (7.4)$$

Thus from (7.2), (7.3), (7.4), and the triangle inequality we see that for  $0 < h < h(t)$ ,

$$\frac{1}{h} \int_t^{t+h} \left| \dot{x}(s) - \sum_i \beta_i(t) g(s, v_i(t)) \right| ds < 3\gamma. \quad (7.5)$$

Let  $I_t = (t, t + h(t))$ ; then  $J = \bigcup_{t \in T} I_t$  covers  $I$  except for a set of measure zero. Let  $E$  be an open set containing this null set such that

$$\int_E (|\dot{x}(s)| + l(s)(M + N \|x\|)) \, dv < \gamma. \quad (7.6)$$

Let  $E, I_{t_1}, \dots, I_{t_N}$  be a finite subcovering of  $I$ . It is assumed that  $t_k < t_{k+1}$  for each  $k$ . Now let  $J_k = I_{t_k} - (\bigcup_{j=1}^{k-1} I_{t_j})$  and let  $J_0 = I - (\bigcup_{k=1}^N J_k) \subset E$ . Now define the function  $\alpha_i(t), u_i(t)$  as follows:

$$\begin{aligned} \alpha_i(t) &= \beta_i(t_k) & \text{for } t \in J_k \quad k = 1, \dots, N; i = 1, \dots, n+1, \\ u_i(t) &= v_i(t_k) & \text{for } t \in J_k \quad k = 1, \dots, N; i = 1, \dots, n+1, \\ \alpha_1(t) &= 1, \quad \alpha_2(t) = \dots = \alpha_{n+1}(t) = 0 & \text{for } t \in J_0, \\ u_i(t) &= u_i & i = 1, \dots, n+1, \quad \text{for } t \in J_0, \end{aligned}$$

where the  $u_i$  are arbitrary fixed elements of  $U$ . Then,

$$\begin{aligned} \int_I \left| \dot{x}(s) - \sum_i \alpha_i(s) g(s, u_i(s)) \right| \, ds &\leq \int_{J_0} \left( |\dot{x}(s)| + \sum_i \alpha_i(s) |g(s, u_i(s))| \right) \, ds \\ &\quad + \sum_k \int_{J_k} \left| \dot{x}(s) - \sum_i \alpha_i(s) g(s, u_i(s)) \right| \, ds \\ &\leq \gamma + 3\gamma \sum_k \text{mes } |J_k| \leq \gamma + 3\gamma \text{mes } (I) \end{aligned}$$

by (7.5) and (7.6). By taking  $\gamma < \epsilon/(1 + 3 \text{mes}(I))$ , we obtain (7.1).

**THEOREM 7.1.**  $\mathcal{R}$  is closed in  $\mathcal{C}$ .

*Proof.* Let  $x \in \mathcal{R}$  so that  $\dot{x}(t) \in G(x(t), t)$  a.e. in  $I$ . But then from the fact that  $G(x, t)$  is the convex closure of  $F(x, t)$  and from condition (iv) of Section 3, we see that

$$|\dot{x}(t)| \leq l(t)(M + N \|x(t)\|) \quad \text{a.e. in } I. \quad (7.7)$$

Therefore the same argument as in Lemma 3.1 shows that there exist  $B < \infty$  such that  $\|x\| \leq B$  for  $x \in \mathcal{R}$ .

Now let  $\{x_n\}$  be a sequence in  $\mathcal{R}$  such that  $x_n \rightarrow \tilde{x}$  for some  $\tilde{x}$  in  $\mathcal{C}$ . It must be shown that (i)  $\tilde{x}$  is absolutely-continuous and (ii)  $\dot{\tilde{x}}(t) \in G(\tilde{x}(t), t)$  a.e. in  $I$ . Because of (7.7), given any  $\epsilon > 0$ , there is a  $\delta > 0$  such that for every  $x$  in  $\mathcal{R}$  and every finite sequence  $t_1 < t'_1 \leq t_2 < t'_2 < \dots \leq t_m < t'_m$  in  $I$  with  $\sum_i |t'_i - t_i| < \delta$ ,

$$\sum_i |x(t'_i) - x(t_i)| \leq \sum_i \int_{t_i}^{t'_i} |\dot{x}(s)| \, ds \leq \sum_i \int_{t_i}^{t'_i} l(s)(M + NB) \, ds \leq \epsilon. \quad (7.8)$$

Therefore,

$$\begin{aligned} \sum_i |\tilde{x}(t'_i) - \tilde{x}(t_i)| &\leq \sum_i |\tilde{x}(t'_i) - x_n(t'_i)| + \sum_i |x_n(t'_i) - x_n(t_i)| \\ &\quad + \sum_i |x_n(t_i) - \tilde{x}(t_i)|. \end{aligned}$$

The second term on the right-hand side is less than  $\epsilon$  by (7.8), and the remaining terms can be made arbitrarily small by choosing  $n$  large since  $\|x_n - \tilde{x}\| \rightarrow 0$ . Therefore,

$$\sum_i |\tilde{x}(t'_i) - \tilde{x}(t_i)| \leq \epsilon,$$

so that  $\tilde{x}$  is absolutely-continuous.

Next it will be shown that the functions  $\dot{x}_n$  and  $\dot{x}$  satisfy the hypothesis of Lemma 7.2. Clearly, from (7.7) the  $\dot{x}_n$  are uniformly bounded by an integrable function and  $\dot{x}$  is integrable. For  $\gamma > 0$  there is  $\delta > 0$  such that

$$\int_A (|\dot{x}(s)| + |\dot{x}_n(s)|) ds < \frac{\gamma}{3}$$

for every  $n$  and every measurable set  $A$  of  $I$  of measure less than  $\delta$ . Let  $E$  be any measurable subset of  $I$ . Let  $J$  be a finite union of intervals  $(t_k, t'_k)$ ,  $k = 1, \dots, p$ , such that measure  $(E - J) < \delta$ , and let  $N$  be so large that  $\|\tilde{x} - x_n\| < (\gamma/3p)$  for  $n > N$ . Then for  $n > N$ ,

$$\begin{aligned} \left| \int_E (\dot{x}_n(s) - \dot{x}(s)) ds \right| &\leq \int_{E-J} |\dot{x}_n(s) - \dot{x}(s)| ds \\ &\quad + \sum_k \left| \int_{t_k}^{t'_k} (\dot{x}_n(s) - \dot{x}(s)) ds \right| \\ &\leq \frac{\gamma}{3} + \sum_k (|x_n(t'_k) - \tilde{x}(t'_k)| + |x_n(t_k) - \tilde{x}(t_k)|) \\ &\leq \gamma. \end{aligned}$$

Hence

$$\int_E \dot{x}_n(s) ds \rightarrow \int_E \dot{x}(s) ds$$

for every measurable subset  $E$  of  $I$ , and therefore for each  $y \in R^n$ ,

$$\int_E \langle y, \dot{x}_n(s) \rangle ds \rightarrow \int_E \langle y, \dot{x}(s) \rangle ds.$$

By Lemma 7.2, for each  $y \in R^n$ ,

$$\overline{\lim}_n \langle y, \dot{x}_n(t) \rangle \geq \langle y, \dot{x}(t) \rangle \geq \underline{\lim}_n \langle y, \dot{x}_n(t) \rangle \quad (7.9)$$

a.e. in  $I$ .

Since the function  $f(x, t, u)$  is continuous in  $x$  for fixed  $u, t$ , the set function  $G(x, t)$  is continuous in  $x$  for fixed  $t$ . Hence for every fixed  $t$  and every  $y \in R^n$ , since  $\|x_n - \hat{x}\| \rightarrow 0$ ,

$$\max_{z \in G(x_n(t), t)} \langle y, z \rangle \rightarrow \max_{z \in G(\hat{x}(t), t)} \langle y, z \rangle,$$

and

$$\min_{z \in G(x_n(t), t)} \langle y, z \rangle \rightarrow \min_{z \in G(\hat{x}(t), t)} \langle y, z \rangle. \quad (7.10)$$

Hence from (7.9) and (7.10), for each  $y \in R^n$  we have

$$\max_{z \in G(\hat{x}(t), t)} \langle y, z \rangle \geq \langle y, \dot{x}(t) \rangle \geq \min_{z \in G(\hat{x}(t), t)} \langle y, z \rangle$$

a.e. in  $I$ . By Lemma 7.1,  $\dot{x}(t) \in G(\hat{x}(t), t)$  a.e. in  $I$  and the theorem is proved.

**LEMMA 7.4.** *Let  $u$  be an admissible control,  $X$  a compact subset of  $R^n$ , and  $\epsilon > 0$ . Then there is a continuous function  $h : X \times I \rightarrow R^n$  such that for every measurable function  $x : I \rightarrow X$ ,*

$$\int_I |f(x(t), t, u(t)) - h(x(t), t)| dt < \epsilon. \quad (7.11)$$

*Proof.* From Section 3,

$$|f(x, t, u) - f(x', t, u)| \leq k(t) |x - x'|$$

for  $u$  in  $U$  and  $x, x'$  in  $R^n$ . Let

$$\delta = \frac{1}{2} \epsilon \left( \int_I k(t) dt \right)^{-1}.$$

Let  $V_1, \dots, V_p$  be a finite open covering of  $X$  such that the diameter of these sets is less than  $\delta$ . Therefore, if  $x, x'$  are in  $V_j$  then  $|x - x'| < \delta$ , so that

$$\int_I |f(x, t, u(t)) - f(x', t, u(t))| dt \leq \delta \int_I K(t) dt = \frac{\epsilon}{2}. \quad (7.12)$$

For each  $j$ , let  $x_j$  be a fixed point in  $V_j$  and let  $h_j : I \rightarrow R^n$  be continuous functions such that

$$\int_I |f(x_j, t, u(t)) - h_j(t)| dt < \frac{\epsilon}{2p}, \quad (7.13)$$

where  $p$  is the integer determined above by  $\delta$ . Now let  $\varphi_1(x), \dots, \varphi_p(x)$  for the partition of unity on  $X$  with respect to the  $\{V_j\}$ ; i.e., the  $\varphi_j$  are continuous real-valued functions on  $X$  with the property that

$$0 \leq \varphi_j(x) \leq 1; \quad \varphi_j(x) = 0 \text{ } x \notin V_j, \quad \sum_j \varphi_j(x) \equiv 1. \quad (7.14)$$

Then define,

$$h(x, t) = \sum_j h_j(t) \varphi_j(x).$$

Now let  $x : I \rightarrow X$  be any measurable function and consider

$$H(t) = \sum_{j=1}^p \varphi_j(x(t)) |x(t) - x_j|.$$

If  $|x(t) - x_j| > \delta$  then  $x(t) \notin V_j$  and so  $\varphi_j(x(t)) = 0$ . If  $|x(t) - x_j(t)| \leq \delta$  then  $\varphi_j(x(t)) |x(t) - x_j| \leq \delta \varphi_j(x(t))$ . Thus for all  $t \in I$ ,  $\varphi_j(x(t)) |x(t) - x_j| \leq \delta \varphi_j(x(t))$  and so by (7.14)

$$H(t) = \sum_{j=1}^p \varphi_j(x(t)) |x(t) - x_j| \leq \delta \sum_{j=1}^p \varphi_j(x(t)) \leq \delta. \quad (7.15)$$

Then by (7.15) and (7.12),

$$\begin{aligned} & \int_I \left| f(x(t), t, u(t)) - \sum_j \varphi_j(x(t)) f(x_j, t, u(t)) \right| dt \\ & \leq \int_I \sum_j \varphi_j(x(t)) |f(x(t), t, u(t)) - f(x_j, t, u(t))| dt \\ & \leq \int_I \sum_j \varphi_j(x(t)) |x(t) - x_j| k(t) dt \\ & = \int_I H(t) k(t) dt \leq \delta \int_I k(t) dt = \frac{1}{2} \epsilon. \end{aligned}$$

By (7.13) and (7.14),

$$\begin{aligned} & \int_I \left| \sum \varphi_j(x(t)) \{f(x_j, t, u(t)) - h_j(t)\} \right| dt \\ & \leq \sum_j \int_I |f(x_j, t, u(t)) - h_j(t)| dt < \sum_{j=1}^p \frac{\epsilon}{2p} \\ & < \frac{\epsilon}{2}. \end{aligned}$$

Now by definition of  $h(x, t)$  and the last two results

$$\begin{aligned} & \int_I |f(x(t), t, u(t)) - h(x(t), t)| dt \\ & = \int_I \left| f(x(t), t, u(t)) - \sum_j \varphi_j(x(t)) f(x_j, t, u(t)) \right. \\ & \quad \left. + \sum_j \varphi_j(x(t)) f(x_j, t, u(t)) - \sum_j \varphi_j(x(t)) h_j(t) \right| dt \\ & \leq \int_I \left| f(x(t), t, u(t)) - \sum_j \varphi_j(x(t)) f(x_j, t, u(t)) \right| dt \\ & \quad + \int_I \left| \sum_j \varphi_j(x(t)) \{f(x_j, t, u(t)) - h_j(t)\} \right| dt \\ & < \epsilon. \end{aligned}$$

LEMMA 7.5. For  $i = 1, \dots, n-1$ , let  $u_i$  be admissible controls and let  $\alpha_i : I \rightarrow R^1$  be measurable functions with  $\alpha_i(t) \geq 0$ ,

$$\sum_i \alpha_i(t) \equiv 1.$$

Let  $x : I \rightarrow R^n$  be any continuous function. Then, for each  $\epsilon > 0$  there is an admissible control  $u_\epsilon$  such that for every  $t_1, t_2$  in  $I$ ,

$$\left| \int_{t_1}^{t_2} \left( \sum_i \alpha_i(s) f(x(s), s, u_\epsilon(s)) - f(x(s), s, u_\epsilon(s)) \right) ds \right| < \epsilon.$$

*Proof.* Let the range of the function  $x$  be the compact subset  $X$  of  $R^n$ , and let  $\gamma > 0$  be fixed. By Lemma 7.4, for each  $i$  there exists a continuous function  $h_i : X \times I \rightarrow R^n$  such that

$$\int |f(x(t), t, u_i(t)) - h_i(x(t), t)| dt < \gamma. \quad (7.16)$$

Let  $\delta_1 > 0$  be such that for every  $x$  in  $X$  and  $t, t'$  in  $I$  with  $|t - t'| < \delta_1$

$$|h_i(x, t) - h_i(x, t')| < \gamma \quad (7.17)$$

for each  $i$ . Let  $\delta_2 > 0$  be such that for every measurable subset  $A$  of  $I$  with measure less than  $\delta_2$ ,

$$\int_A (M + NB) l(t) dt < \gamma. \quad (7.18)$$

Here  $B$  is the uniform bound of  $\mathcal{B}$ . Let  $\delta = \min\{\delta_1, \delta_2\}$  and let  $a = t_0 < t_1 < \dots < t_p = b$  with  $(t_{i+1} - t_i) < \delta$  for each  $i$ . Let  $I_k = [t_{k-1}, t_k]$  and subdivide each interval  $I_k$  into  $n + 1$  subinterval  $I_{k,1}, I_{k,2}, \dots, I_{k,n+1}$  such that the measure of  $I_{k,j}$  is

$$|I_{k,j}| = \int_{I_k} \alpha_j(t) dt. \quad (7.19)$$

Now define the admissible control  $\tilde{u}$ , by

$$\tilde{u}(t) = u_j(t) \text{ whenever } t \in I_{k,j} \text{ for some } j, k. \quad (7.20)$$

For convenience denote  $\tilde{f}_i(t) = f_i(x(t), t, u_i(t))$ ,  $\tilde{f}(t) = f(x(t), t, \tilde{u}(t))$ , and  $\tilde{h}_i(t) = h_i(x(t), t)$ . Then from (7.20),

$$\int_{I_k} \tilde{f}(t) dt = \sum_j \int_{I_{k,j}} \tilde{f}_j(t) dt. \quad (7.21)$$

Let  $r_1, r_2$  be any integers with  $1 \leq r_1 \leq r_2 \leq p$ . Then from (7.21),

$$\begin{aligned} \Delta(r_1, r_2) &= \sum_{k=r_1}^{r_2} \left| \int_{I_k} \sum_i \alpha_i \tilde{f}_i - \tilde{f} \right| = \sum_{k=r_1}^{r_2} \left| \int_{I_k} \sum_i \alpha_i \tilde{f}_i - \sum_i \int_{I_{k,i}} \tilde{f}_i \right| \\ &= \sum_{k=r_1}^{r_2} \left| \int_{I_k} \sum_i \alpha_i (\tilde{f}_i - \tilde{h}_i) + \int_{I_k} \sum_i \alpha_i \tilde{h}_i - \sum_i \int_{I_{k,i}} \tilde{h}_i \right. \\ &\quad \left. - \sum_i \int_{I_{k,i}} (\tilde{f}_i - \tilde{h}_i) \right| \\ &\leq \left( \sum_{k=r_1}^{r_2} \sum_i \int_{I_k} \alpha_i |\tilde{f}_i - \tilde{h}_i| \right) + \left( \sum_{k=r_1}^{r_2} \sum_i \int_{I_{k,i}} |\tilde{f}_i - \tilde{h}_i| \right) \\ &\quad + \left( \sum_{k=r_1}^{r_2} \left| \int_{I_k} \sum_i \alpha_i \tilde{h}_i - \sum_i \int_{I_{k,i}} \tilde{h}_i \right| \right). \end{aligned} \quad (7.22)$$



Consider the first term of the last bound in (7.22); by (7.16),

$$\begin{aligned} \sum_{k=r_1}^{r_2} \sum_i \int_{I_k} \alpha_i |\tilde{f}_i - \tilde{h}_i| &\leq \sum_i \sum_{k=r_1}^{r_2} \int_{I_k} |\tilde{f}_i - \tilde{h}_i| \\ &\leq \sum_i \int_I |\tilde{f}_i - \tilde{h}_i| \leq (n+1)\gamma. \end{aligned} \quad (7.23)$$

Similarly by (7.16), the second term is bounded by  $(n+1)\gamma$ . The third term can be written as

$$\begin{aligned} \sum_{k=r_1}^{r_2} \left| \int_{I_k} \sum_i \alpha_i \tilde{h}_i - \sum_i \int_{I_{ki}} \tilde{h}_i \right| \\ = \sum_{k=r_1}^{r_2} \left| \int_{I_k} \sum_i \alpha_i(t) (\tilde{h}_i(t) - \tilde{h}_i(t_k)) - \sum_i \int_{I_{ki}} (\tilde{h}_i(t) - \tilde{h}_i(t_k)) \right| \\ \leq \sum_{k=r_1}^{r_2} \left( \int_{I_k} \sum_i \alpha_i(t) |\tilde{h}_i(t) - \tilde{h}_i(t_k)| + \sum_i \int_{I_{ki}} |\tilde{h}_i(t) - \tilde{h}_i(t_k)| \right) \\ \leq \sum_{k=r_1}^{r_2} \left( \gamma \int_{I_k} \sum_i \alpha_i(t) dt + \gamma \sum_i \int_{I_{ki}} dt \right) \leq \sum_{k=r_1}^{r_2} 2\gamma \text{mes}(I_k) \\ \leq 2\gamma \text{mes}(I). \end{aligned} \quad (7.24)$$

The third-last bound in (7.24) follows from (7.17). Using the above bound, we see from (7.22) that,

$$\Delta(r_1, r_2) \leq 2\gamma(n+1 + |I|). \quad (7.25)$$

Now let  $t_1$  and  $t_2$  be arbitrary points in  $I$  with  $t_1 \leq t_2$ . Let  $r_1$  and  $r_2$  be integers such that

$$\begin{aligned} \delta(t_1, t_2) &= \left| \int_{t_1}^{t_2} \left( \sum_i \alpha_i \tilde{f}_i - \tilde{f} \right) \right| \\ &\leq \sum_{k=r_1}^{r_2} \left| \int_{I_k} \left( \sum_i \alpha_i \tilde{f}_i - \tilde{f} \right) \right| + \left| \int_{\tilde{I}_1} \left( \sum_i \alpha_i \tilde{f}_i - \tilde{f} \right) \right| + \left| \int_{\tilde{I}_2} \left( \sum_i \alpha_i \tilde{f}_i - \tilde{f} \right) \right| \end{aligned} \quad (7.26)$$

where  $\tilde{I}_1$  is a subinterval of  $I_{r_1-1}$  and  $\tilde{I}_2$  is a subinterval of  $I_{r_2+1}$ . Then by (7.18),

$$\begin{aligned} \left| \int_{\tilde{I}_1} \left( \sum_i \alpha_i \tilde{f}_i - \tilde{f} \right) \right| &\leq \int_{\tilde{I}_1} \left( \sum_i \alpha_i |\tilde{f}_i| + |\tilde{f}| \right) \\ &\leq \int_{\tilde{I}_1} (M + NB) l(t) \left( \sum_i \alpha_i + 1 \right) < 2\gamma. \end{aligned} \quad (7.27)$$

Similarly,

$$\left| \int_{\tilde{I}_2} \left( \sum_i \alpha_i \tilde{f}_i - \tilde{f} \right) \right| < 2\gamma. \quad (7.28)$$

From (7.26), (7.27), (7.28), and (7.25) we see that

$$\begin{aligned} \delta(t_1, t_2) &\leq \Delta(r_1, r_2) + 4\gamma \leq 2\gamma(n + 1 + |I|) + 4\gamma \\ &= 2\gamma(n + 3 + |I|). \end{aligned}$$

Therefore, for  $\gamma = (\mathcal{E}/2)(n + 3 + |I|)^{-1}$  the function  $\tilde{u}$  defined in (7.2) satisfies the assertion of the lemma.

THEOREM 7.2.  $\mathcal{R} = \mathcal{F}$ .

*Proof.* By definition,  $\mathcal{R} \supset \mathcal{F}$  and by Theorem 7.1  $\mathcal{R}$  is closed so that it suffices to show that  $\mathcal{R} \subset \mathcal{F}$ . To this end, let  $x$  in  $\mathcal{R}$  and  $\epsilon > 0$  be given. Let

$$\gamma = \frac{\mathcal{E}}{2} \left[ \exp \left( \int_I k(t) dt \right) \right]^{-1} \quad (7.29)$$

where  $k$  is the function of condition (iii) of Section 3. By Lemma 7.3, there exist measurable functions  $\alpha_i$ ,  $i = 1, \dots, n + 1$  with  $\alpha_i(t) \geq 0$  and  $\sum \alpha_i(t) \equiv 1$ , and there exist admissible controls  $u_1, \dots, u_{n+1}$  such that for all  $t$  in  $I$

$$\int_a^t \left| \dot{x}(s) - \sum_i \alpha_i(s) f(x(s), s, u_i(s)) \right| ds < \gamma. \quad (7.30)$$

By Lemma 7.5, there exists an admissible control  $u_\gamma$  such that for all  $t$  in  $I$

$$\left| \int_a^t \left( \sum_i \alpha_i(s) f(x(s), s, u_i(s)) - f(x(s), s, u_\gamma(s)) \right) ds \right| < \gamma. \quad (7.31)$$

Let  $\gamma$  in  $\mathcal{F}$  be defined by

$$\dot{y}(t) = f(y(t), t, u_\gamma(t)) \quad (7.32)$$

with

$$y(a) = x(a).$$

Then from (7.30), (7.31), and condition (iii) of Section 3,

$$\begin{aligned}
 |x(t) - y(t)| &= \left| \int_a^t \dot{x}(s) ds - \int_a^t f(y(s), s, u_i(s)) ds \right| \\
 &\leq \left| \int_a^t \left( \dot{x}(s) - \sum_i \alpha_i(s) f(x(s), s, u_i(s)) \right) ds \right| \\
 &\quad + \left| \int_a^t \left( \sum_i \alpha_i(s) f(x(s), s, u_i(s)) - f(x(s), s, u_r(s)) \right) ds \right| \\
 &\quad + \left| \int_a^t (f(x(s), s, u_r(s)) - f(y(s), s, u_r(s))) ds \right| \\
 &\leq 2\gamma + \int_a^t k(s) |x(s) - y(s)| ds.
 \end{aligned}$$

By Gronwall's lemma, the last estimate implies that for each  $t$  in  $I$ ,

$$|x(t) - y(t)| \leq 2\gamma \exp \left( \int_a^t k(s) ds \right) < \epsilon.$$

This completes the proof of Theorem 7.2.

Combining Theorems 6.1 and 7.2 gives

THEOREM 7.3.  $\mathcal{R} = \mathcal{F} = \mathcal{M}$ .

## APPENDIX

LEMMA A1. For each  $x$  in  $R^n$  let  $T(x)$  be a subset of  $R^n$  such that the mapping  $x \rightarrow T(x)$  is continuous; i.e., for  $x$  in  $R^n$  and  $\epsilon > 0$  there is a  $\delta > 0$  such that for  $|x - y| < \delta$

$$T(x) \subset S_\epsilon[T(y)] \tag{A-1}$$

and

$$T(y) \subset S_\epsilon[T(x)]. \tag{A-2}$$

Then for every bounded set  $A$  in  $R^n$

$$\overline{\bigcup_{x \in A} T(x)} = \bigcup_{x \in A} \overline{T(x)}. \tag{A-3}$$

*Proof.* Let  $y \in \overline{\bigcup_{x \in A} T(x)}$ ; then there is a sequence  $y_n \in \bigcup_{x \in A} T(x)$  such that  $y_n \rightarrow y$ . For each  $n$  let  $x_n$  in  $A$  be such that  $y_n \in T(x_n)$ . Taking sub-

sequences, if necessary, it can be assumed that  $x_n$  converges to  $x$  in  $\bar{A}$ . Given  $\epsilon > 0$  let  $N$  be so large that, for  $n > N$

$$T(x_n) \subset S_\epsilon[T(x)]$$

and

$$|y_n - y| < \epsilon,$$

so that,

$$\rho(y, T(x)) \leq 2\epsilon.$$

Since  $\epsilon$  is arbitrary, this implies that  $y \in \overline{T(x)}$  and hence

$$\bigcup_{x \in A} \overline{T(x)} \subset \bigcup_{x \in \bar{A}} \overline{T(x)}.$$

On the other hand, let  $y \in \bigcup_{x \in \bar{A}} \overline{T(x)}$ ; then there is  $x \in \bar{A}$  such that  $y \in \overline{T(x)}$ . Let  $x_n$  be a sequence in  $A$  converging to  $x$ , and let  $y_n$  be a sequence in  $T(x)$  converging to  $y$ . Let  $\epsilon > 0$ , and let  $N$  be so large that,

$$T(x) \subset S_\epsilon[T(x_n)]$$

and

$$|y - y_n| < \epsilon$$

for  $n > N$ . Therefore  $\rho(y, \bigcup_{x \in A} T(x)) < 2\epsilon$  and since  $\epsilon$  is arbitrary  $y \in \overline{\bigcup_{x \in A} T(x)}$  and the lemma is proved.

#### REFERENCES

1. ROXIN, E., Dynamical systems with inputs, in "Proceedings of the Symposium on System Theory," pp. 105-144. Polytechnic Institute of Brooklyn, New York, 1965.
2. ROXIN, E., Stability in general control systems. *J. Differential Eqs.* (1) (1965), 115-150.
3. ROXIN, E., On generalized dynamical systems defined by contingent equations. *J. Differential Eqs.* 1 (1965), 188-205.
4. WARGA, J., Relaxed variational problems. *J. Math. Anal. Appl.* 4 (1962), 111-128.
5. SANSONE, G. AND CONTI, R., "Nonlinear Differential Equations." MacMillan (Pergamon), New York, 1964.
6. CULLUM, J., Continuous optimal control problems with phase space constraints. Doctoral dissertation, University of California, Berkeley, 1965.